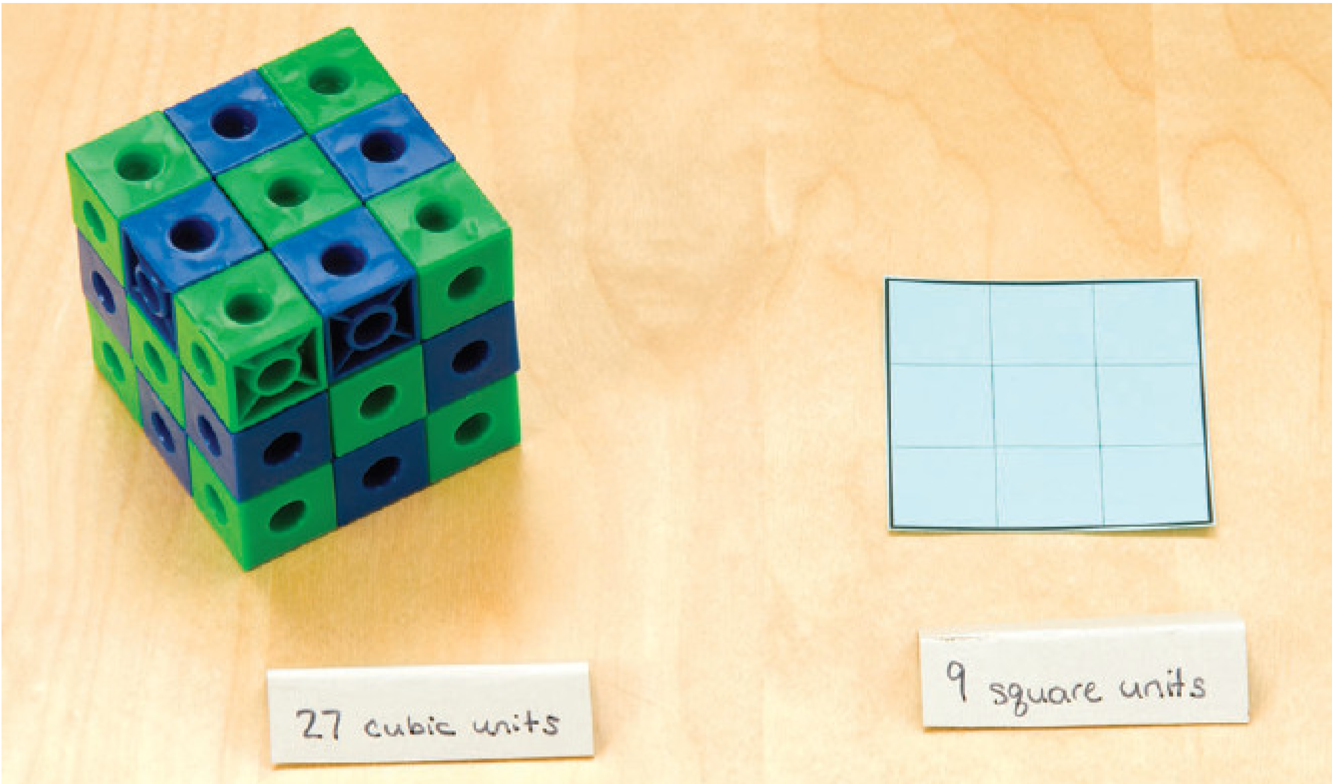
4.1 Math Lab: Estimating Roots

LESSON FOCUS: Explore decimal representations of different roots of numbers.



Make Connections

Since $3^2 = 9$, 3 is a square root of 9.

We write: $3 = \sqrt{9}$

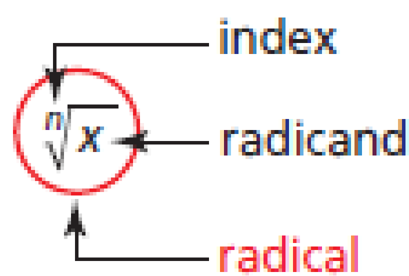
Since $3^3 = 27$, 3 is the cube root of 27.

We write: $3 = \sqrt[3]{27}$

Since $3^4 = 81$, 3 is a fourth root of 81.

We write: $3 = \sqrt[4]{81}$

How would you write 5 as a square root?
A cube root? A fourth root?



Construct Understanding

TRY THIS

Work with a partner.

You will need a calculator to check your estimates.

A. Write the two consecutive perfect squares closest to 20. Estimate the value of $\sqrt{20}$. Square your estimate. Use this value to revise your estimate. Keep revising your estimate until the square of the estimate is within 1 decimal place of 20.

B. Write the two consecutive perfect cubes closest to 20. Estimate the value of $\sqrt[3]{20}$. Cube your estimate. Use this value to revise your estimate. Keep revising your estimate until the cube of the estimate is within 1 decimal place of 20.

C. Write the two consecutive perfect fourth powers closest to 20. Use a strategy similar to that in Steps A and B to estimate a value for $\sqrt[4]{20}$.

D. Copy and complete this table. Use the strategies from Steps A to C to determine the value of each radical.

Radical	Value	Is the Value Exact or Approximate?
$\sqrt{16}$	4	Exact
$\sqrt{27}$	5.1962	Approximate
$\sqrt{\frac{16}{81}}$	$\frac{4}{9}$ or $0.\overline{4}$	Exact
$\sqrt{0.64}$		
$\sqrt[3]{16}$		
$\sqrt[3]{27}$		
$\sqrt[3]{\frac{16}{81}}$		
$\sqrt[3]{0.64}$		
$\sqrt[3]{-0.64}$		
$\sqrt[4]{16}$		
$\sqrt[4]{27}$		
$\sqrt[4]{\frac{16}{81}}$		
$\sqrt[4]{0.64}$		

Choose 3 different radicals.
Extend then complete the table for these radicals.

E. How can you tell if the value of a radical is a rational number? What strategies can you use to determine the value of the radical?

F. How can you tell if the value of a radical is *not* a rational number? What strategies can you use to estimate the value of the radical?

Homework: Page 206 #1-6

4.2 Irrational Numbers

LESSON FOCUS: Identify and order irrational numbers.

The room below the rotunda in the Manitoba Legislative Building is the Pool of the Black Star. It has a circular floor.

Make Connections

The formulas for the area and circumference of a circle involve π, which is not a rational number because it cannot be written as a quotient of integers.

What other numbers are not rational?

Construct Understanding

TRY THIS

Work with a partner.

These are rational numbers.	These are not rational numbers.
$\sqrt{100}$ $\sqrt{0.25}$ $\sqrt[3]{8}$ 0.5	$\sqrt{0.24}$ $\sqrt[3]{9}$ $\sqrt{2}$
$\frac{5}{6}$ $\sqrt{\frac{9}{64}}$ 0.8^2 $\sqrt[5]{-32}$	$\sqrt{\frac{1}{3}}$ $\sqrt[4]{12}$

A. How are radicals that are rational numbers different from radicals that are not rational numbers?

B. Which of these radicals are rational numbers?
Which are not rational numbers? How do you know?

$\sqrt{1.44}$, $\sqrt{\frac{64}{81}}$, $\sqrt[3]{-27}$, $\sqrt{\frac{4}{5}}$, $\sqrt{5}$

C. Write 3 other radicals that are rational numbers.
Why are they rational?

D. Write 3 other radicals that are not rational numbers.
Why are they not rational?

Irrational Numbers

An **irrational number** *cannot* be written in the form $\frac{m}{n}$, where m and n are integers, $n \neq 0$. The decimal representation of an irrational number neither terminates nor repeats.

If x is not a perfect square. Then the number $\sqrt{x}$ is irrational.
If x is not a perfect cube. Then the number $\sqrt[3]{x}$ is irrational.
Etc.

Example 1: Classifying Numbers
Tell whether each number is rational or irrational.
Explain how you know.

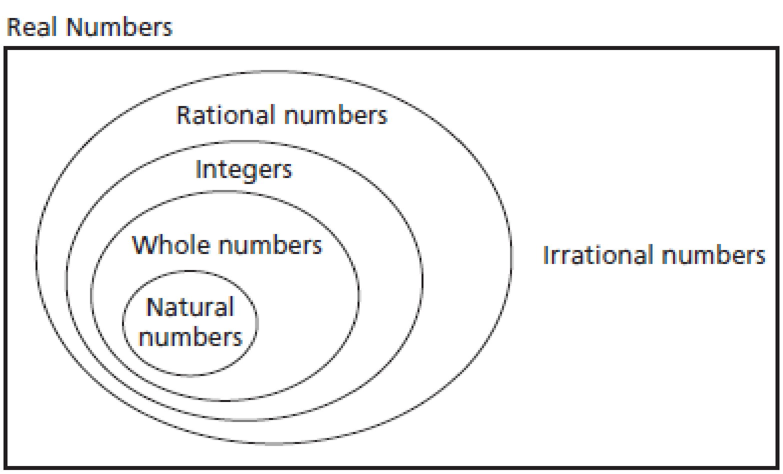
a) $-\frac{3}{5}$ **b)** $\sqrt{14}$ **c)** $\sqrt[3]{\frac{8}{27}}$

CHECK YOUR UNDERSTANDING
Tell whether each number is rational or irrational.
Explain how you know.

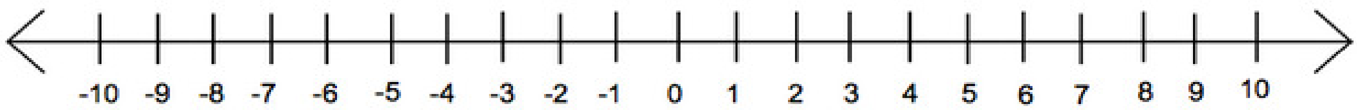
a) $\sqrt{\frac{49}{16}}$ **b)** $\sqrt[3]{-30}$ **c)** $\sqrt{1.21}$ [Answers: a) rational b) irrational c) rational]

Together, the rational numbers and irrational numbers form the set of **real numbers**.

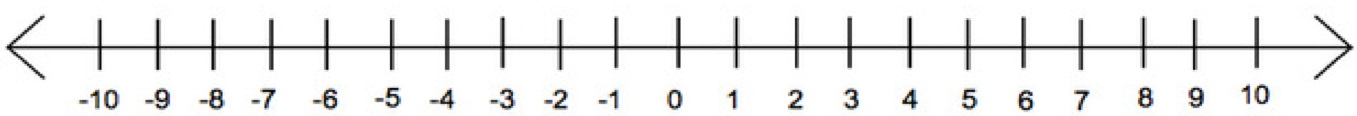
This diagram shows how these number systems are related.



Example 2: Ordering Irrational Numbers on a Number Line
Use a number line to order these numbers from least to greatest.
$\sqrt[3]{13},\ \sqrt{18},\ \sqrt{9},\ \sqrt[4]{27},\ \sqrt[3]{-5}$



CHECK YOUR UNDERSTANDING
Use a number line to order these numbers from least to greatest.
$\sqrt{2},\ \sqrt[3]{-2},\ \sqrt[3]{6},\ \sqrt{11},\ \sqrt[4]{30}$

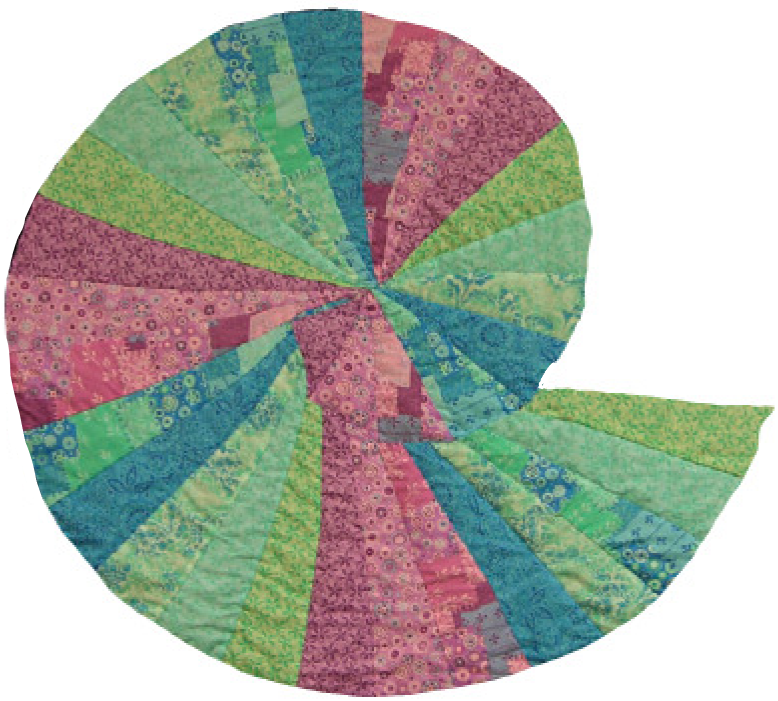


[Answers: $\sqrt[3]{-2},\ \sqrt{2},\ \sqrt[3]{6},\ \sqrt[4]{30},\ \sqrt{11}$]

Homework: Page 211 #3-5, 7, 11, 12, 14, 15, 17-19

4.3 Mixed and Entire Radicals

LESSON FOCUS: Express an entire radical as a mixed radical, and vice versa.



This quilt represents a Pythagorean spiral. The smallest triangle is a right isosceles triangle with legs 1 unit long.

Make Connections

We can name the fraction $\frac{3}{12}$ in many different ways:

$\frac{1}{4}$ $\quad \frac{5}{20}$ $\quad \frac{30}{120}$ $\quad \frac{100}{400}$

How do you show that each fraction is equivalent to $\frac{3}{12}$?
Why is $\frac{1}{4}$ the simplest form of $\frac{3}{12}$?

Just as with fractions, equivalent expressions for any number have the same value.

- $\sqrt{16 \cdot 9}$ is equivalent to $\sqrt{16} \cdot \sqrt{9}$ because:

  $\sqrt{16 \cdot 9} = \sqrt{144} = 12$ and $\sqrt{16} \cdot \sqrt{9} = 4 \cdot 3 = 12$

- Similarly, $\sqrt[3]{8 \cdot 27}$ is equivalent to $\sqrt[3]{8} \cdot \sqrt[3]{27}$ because:

  $\sqrt[3]{8 \cdot 27} = \sqrt[3]{216} = 6$ and $\sqrt[3]{8} \cdot \sqrt[3]{27} = 2 \cdot 3 = 6$

Multiplication Property of Radicals

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$,
where n is a natural number, and a and b are real numbers

Example 1: Simplifying Radicals Using Prime Factorization
Simplify each radical.

a) $\sqrt{80}$ **b)** $\sqrt[3]{144}$ **c)** $\sqrt[4]{162}$

CHECK YOUR UNDERSTANDING
Simplify each radical.

a) $\sqrt{63}$ **b)** $\sqrt[3]{108}$ **c)** $\sqrt[4]{128}$

[Answers: *a*) $3\sqrt{7}$, *b*) $3\sqrt[3]{4}$, *c*) $2\sqrt[4]{8}$]

Example 2: Writing Radicals in Simplest Form
Write each radical in simplest form, if possible.

a) $\sqrt{26}$ **b)** $\sqrt[3]{40}$ **c)** $\sqrt[4]{32}$

CHECK YOUR UNDERSTANDING
Write each radical in simplest form, if possible.

a) $\sqrt{30}$ **b)** $\sqrt[3]{32}$ **c)** $\sqrt[4]{48}$

[Answers: *a*) cannot be simplified, *b*) $2\sqrt[3]{4}$, *c*) $2\sqrt[4]{3}$]

Radicals of the form $\sqrt[n]{x}$ such as $\sqrt{80}$, $\sqrt[3]{144}$, and $\sqrt[4]{162}$ are **entire radicals.**

Radicals of the form $a\sqrt[n]{x}$ such as $4\sqrt{5}$, $2\sqrt[3]{18}$, and $3\sqrt[4]{2}$ are **mixed radicals.** Entire radicals were rewritten as mixed radicals in *Examples 1* and *2.*

Any number can be written as the square root of its square; for example, $2 = \sqrt{2 \cdot 2}$, $3 = \sqrt{3 \cdot 3}$, $4 = \sqrt{4 \cdot 4}$, and so on. Similarly, any number can be written as the cube root of its cube, or the fourth root of its perfect fourth power. We use this strategy to write a mixed radical as an entire radical.

Example 3: Writing Mixed Radicals as Entire Radicals

Write each mixed radical as an entire radical.

a) $4\sqrt{3}$ **b)** $3\sqrt[3]{2}$ **c)** $2\sqrt[5]{2}$

CHECK YOUR UNDERSTANDING

Write each mixed radical as an entire radical.

a) $3\sqrt{7}$ **b)** $2\sqrt[3]{4}$ **c)** $2\sqrt[5]{3}$

[Answers: *a*) $\sqrt{63}$, *b*) $\sqrt[3]{32}$, *c*) $\sqrt[5]{96}$]

Example 4: Ordering Irrational Numbers

Arrange in order from least to greatest without a calculator.

$5,\ 4\sqrt{2},\ 2\sqrt{6},\ 3\sqrt{3}$

CHECK YOUR UNDERSTANDING

Arrange in order from least to greatest without a calculator.

$3\sqrt{5},\ \sqrt{31},\ 2\sqrt{7},\ 4\sqrt{2}$ [Answers: $2\sqrt{7}$, $\sqrt{31}$, $4\sqrt{2}$, $3\sqrt{5}$

Homework: Page 218 #4, 5, 9-12, 15-18, 20-22

<u>4.4 Fractional Exponents and Radicals</u>

LESSON FOCUS: Relate rational exponents and radicals.

Make Connections

Coffee, tea, and hot chocolate contain caffeine. The expression $100(0.87)^{\frac{1}{2}}$ represents the percent of caffeine left in your body $\frac{1}{2}$ h after you drink a caffeine beverage.

Given that $0.87^1 = 0.87$ and $0.87^0 = 1$, how can you estimate a value for $0.87^{\frac{1}{2}}$?

Construct Understanding

TRY THIS

Work with a partner.

A. Copy then complete each table. Use a calculator to complete the second column.

x	$x^{\frac{1}{2}}$
1	$1^{\frac{1}{2}} =$
4	$4^{\frac{1}{2}} =$
9	
16	
25	

x	$x^{\frac{1}{3}}$
1	
8	
27	
64	
125	

Continue the pattern. Write the next 3 lines in each table.

B. For each table:

- What do you notice about the numbers in the first column? Compare the numbers in the first and second columns. What conclusions can you make?
- What do you think the exponent $\frac{1}{2}$ means? Confirm your prediction by trying other examples on a calculator.
- What do you think the exponent $\frac{1}{3}$ means? Confirm your prediction by trying other examples on a calculator.

C. What do you think $a^{\frac{1}{4}}$ and $a^{\frac{1}{5}}$ mean?
Use a calculator to test your predictions for different values of a.

D. What does $a^{\frac{1}{n}}$ mean? Explain your reasoning.

Powers with Rational Exponents with Numerator 1

When n is a natural number and x is a rational number, $x^{\frac{1}{n}} = \sqrt[n]{x}$

Example 1: Evaluating Powers of the Form $a^{\frac{1}{n}}$
Evaluate each power without using a calculator.

a) $27^{\frac{1}{3}}$ **b)** $0.49^{\frac{1}{2}}$ **c)** $(-64)^{\frac{1}{3}}$ **d)** $\left(\frac{4}{9}\right)^{\frac{1}{2}}$

CHECK YOUR UNDERSTANDING
Evaluate each power without using a calculator.

a) $1000^{\frac{1}{3}}$ **b)** $0.25^{\frac{1}{2}}$ **c)** $(-8)^{\frac{1}{3}}$ **d)** $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

[Answers: a) 10 b) 0.5 c) -2 d) $\frac{2}{3}$]

Powers with Rational Exponents

When m and n are natural numbers, and x is a rational number,

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m \quad \text{and} \quad x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

Example 2: Rewriting Powers in Radical and Exponent Form

a) Write $40^{\frac{2}{3}}$ in radical form in 2 ways. **b)** Write $\sqrt{3^5}$ and $\left(\sqrt[3]{25}\right)^2$ in exponent form.

CHECK YOUR UNDERSTANDING

a) Write $26^{\frac{2}{5}}$ in radical form in 2 ways. **b)** Write $\sqrt{6^5}$ and $\left(\sqrt[4]{19}\right)^3$ in exponent form.

[Answers: a) $\sqrt[5]{26^2}$ **or** $\left(\sqrt[5]{26}\right)^2$ b) $6^{\frac{5}{2}}$, $19^{\frac{3}{4}}$]

Example 3: Evaluating Powers with Rational Exponents and Rational Bases

Evaluate:

a) $0.04^{\frac{3}{2}}$ b) $27^{\frac{4}{3}}$ c) $(-32)^{0.4}$ d) $1.8^{1.4}$

CHECK YOUR UNDERSTANDING

Evaluate:

a) $0.01^{\frac{3}{2}}$ b) $(-27)^{\frac{4}{3}}$ c) $81^{\frac{3}{4}}$ d) $0.75^{1.2}$

[Answers: **a)** 0.001 **b)** 81 **c)** 27 **d)** 0.7080....]

Example 4: Applying Rational Exponents

Biologists use the formula $b = 0.01m^{\frac{2}{3}}$ to estimate the brain mass, *b* kilograms, of a mammal with body mass *m* kilograms. Estimate the brain mass of each animal.

a) a husky with a body mass of 27 kg

b) a polar bear with a body mass of 200 kg

CHECK YOUR UNDERSTANDING

Use the formula $b = 0.01m^{\frac{2}{3}}$ to estimate the brain mass of each animal.

a) a moose with a body mass of 512 kg.

b) a cat with a body mass of 5 kg

[Answers: a) approximately 0.64 kg b) approximately 0.03 kg]

Homework: Page 227 #3-7, 10-15, 18-21

4.5 Negative Exponents and Reciprocals

LESSON FOCUS: Relate negative exponents to reciprocals.

Scientists can calculate the speed of a dinosaur from its tracks. These tracks were found near Grand Cache, Alberta.

Powers with Negative Exponents

When x is any non-zero number and n is a rational number, x^{-n} is the reciprocal of x^n.

That is, $x^{-n} = \frac{1}{x^n}$ and $\frac{1}{x^{-n}} = x^n, x \neq 0$

Why can't x be 0?

Example 1: Evaluating Powers with Negative Integer Exponents
Evaluate each power.

a) 3^{-2} b) $\left(-\frac{3}{4}\right)^{-3}$ c) 0.3^{-4}

CHECK YOUR UNDERSTANDING

Evaluate each power. [Answers: a) $\frac{1}{49}$ b) $\frac{27}{1000}$ c] $-\frac{8}{27}$]

a) 7^{-2} b) $\left(\frac{10}{3}\right)^{-3}$ c) $(-1.5)^{-3}$

We can apply the meaning of rational exponents and negative exponents to evaluate powers with negative rational exponents. For example,

the rational exponent in the power $8^{-\frac{2}{3}}$ indicates these operations at the right.

$$\overset{\text{reciprocal}}{\downarrow}\; 8^{-\frac{2}{3}} \quad \begin{matrix} 2 \leftarrow \text{square} \\ 3 \leftarrow \text{cube root} \end{matrix}$$

Since the exponent $-\frac{2}{3}$ is the product: $(-1)\left(\frac{1}{3}\right)(2)$, and order does not matter when we multiply, we can apply the three operations of reciprocal, cube root, and square in any order.

Example 2: Evaluating Powers with Negative Rational Exponents

Evaluate each power without using a calculator.

a) $8^{-\frac{2}{3}}$

b) $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

CHECK YOUR UNDERSTANDING

Evaluate each power without using a calculator. [Answers: a) $\frac{1}{32}$ b) $\frac{6}{5}$]

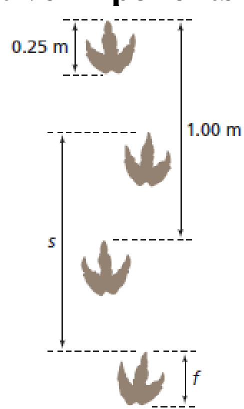
a) $16^{-\frac{5}{4}}$

b) $\left(\frac{25}{36}\right)^{-\frac{1}{2}}$

Example 3: Applying Negative Exponents

Paleontologists use measurements from fossilized dinosaur tracks and the formula $v = 0.155\,s^{\frac{5}{3}}f^{-\frac{7}{6}}$ to estimate the speed at which the dinosaur travelled. In the formula, v is the speed in metres per second, s is the distance between successive footprints of the same foot, and f is the foot length in metres.

Use the measurements in the diagram to estimate the speed of the dinosaur.



CHECK YOUR UNDERSTANDING

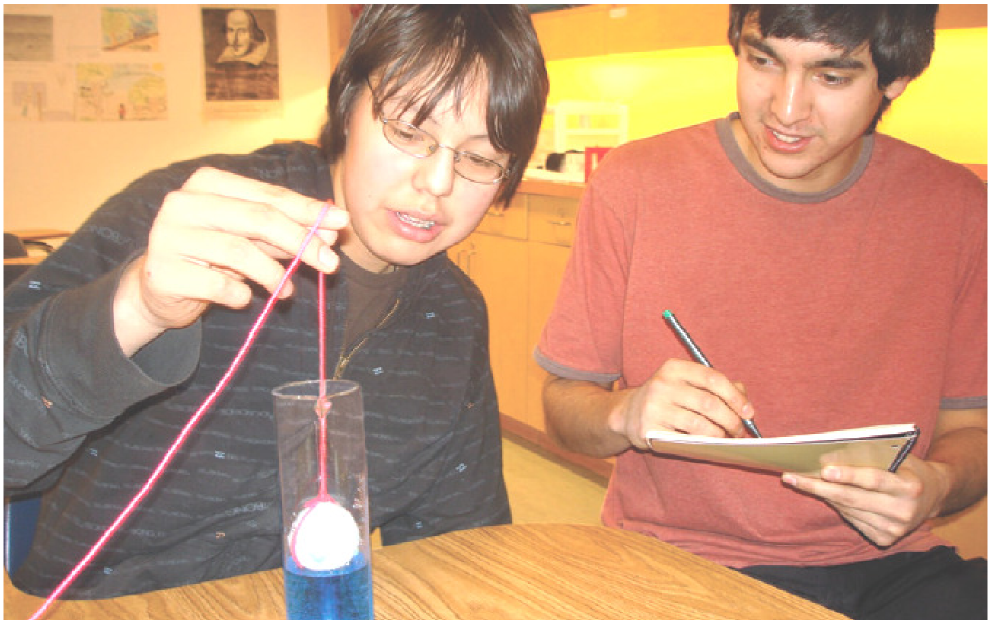
Use the formula $v = 0.155\,s^{\frac{5}{3}}f^{-\frac{7}{6}}$ to estimate the speed of a dinosaur when $s = 1.5$ and $f = 0.3$.

[Answer: approximately 1.2 m/s]

Homework: Page 233 #7–10, 12, 13, 16

4.6 Applying the Exponent Laws

LESSON FOCUS: Apply the exponent laws to simplify expressions.



We can use a measuring cylinder to determine the volume of a sphere. Then we can use the exponent laws to help calculate the radius.

Make Connections

Recall the exponent laws for integer bases and whole number exponents.

Product of powers:	$a^m \cdot a^n = a^{m+n}$
Quotient of powers:	$a^m \div a^n = a^{m-n}, a \neq 0$
Power of a power:	$(a^m)^n = a^{mn}$
Power of a product:	$(ab)^m = a^m b^m$
Power of a quotient:	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$

What other types of numbers could be a base? An exponent?
How would you use the exponent laws to evaluate an expression with these numbers?

Construct Understanding

THINK ABOUT IT

Work on your own.

What is the value of $\left(\frac{a^6b^9}{a^5b^8}\right)^{-2}$ when $a = -3$ and $b = 2$?

Compare strategies with a classmate.
If you used the same strategy, find a different strategy.
Which strategy is more efficient, and why?

Example 1: Simplifying Numerical Expressions with Rational Number Bases
Simplify by writing as a single power.

a) $\left(0.3^{-3}\right)\left(0.3^{5}\right)$ b) $\left[\left(-\frac{3}{2}\right)^{-4}\right]^{2}\left[\left(-\frac{3}{2}\right)^{2}\right]^{3}$ c) $\frac{\left(1.4^{3}\right)\left(1.4^{4}\right)}{1.4^{-2}}$ d) $\left(\frac{7^{\frac{2}{3}}}{7^{\frac{1}{3}}\cdot 7^{\frac{5}{3}}}\right)^{6}$

CHECK YOUR UNDERSTANDING

Simplify by writing as a single power. [Answers: a) 0.8^{-5} b) $\left(-\frac{4}{5}\right)^{14}$ c] 1.5^{10} d) $9^{\frac{1}{4}}$]

a) $0.8^{2}\cdot 0.8^{-7}$ b) $\left[\left(-\frac{4}{5}\right)^{2}\right]^{-3}\div\left[\left(-\frac{4}{5}\right)^{4}\right]^{-5}$ c) $\frac{\left(1.5^{-3}\right)^{-5}}{1.5^{5}}$ d) $\frac{9^{\frac{5}{4}}\cdot 9^{-\frac{1}{4}}}{9^{\frac{3}{4}}}$

Example 2: Simplifying Algebraic Expressions with Integer Exponents
Simplify.

a) $\left(x^3y^2\right)\left(x^2y^{-4}\right)$

b) $\dfrac{10a^5b^3}{2a^2b^{-2}}$

CHECK YOUR UNDERSTANDING

Simplify. [Answers: a) m^6n b) $\dfrac{3x^3}{7y^5}$]

a) $\left(m^4n^{-2}\right)\left(m^2n^3\right)$

b) $\dfrac{6x^4y^{-3}}{14xy^2}$

Example 3: Simplifying Algebraic Expressions with Rational Exponents
Simplify.

a) $\left(8a^3b^6\right)^{\frac{1}{3}}$

b) $\left(x^{\frac{3}{2}}y^2\right)\left(x^{\frac{1}{2}}y^{-1}\right)$

c) $\dfrac{4a^{-2}b^{\frac{2}{3}}}{2a^2b^{\frac{1}{3}}}$

d) $\left(\dfrac{100a}{25a^5b^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$

CHECK YOUR UNDERSTANDING

Simplify. [Answers: a) $125a^6b^3$ b) $\dfrac{x^2}{y}$ c] $\dfrac{4y^3}{x^{\frac{11}{2}}}$ d) $\dfrac{5}{xy^{\frac{3}{2}}}$]

a) $\left(25a^4b^2\right)^{\frac{3}{2}}$

b) $\left(x^3y^{-\frac{3}{2}}\right)\left(x^{-1}y^{\frac{1}{2}}\right)$

c) $\dfrac{12x^{-5}y^{\frac{5}{2}}}{3x^{\frac{1}{2}}y^{-\frac{1}{2}}}$

d) $\left(\dfrac{50x^2y^4}{2x^4y^7}\right)^{\frac{1}{2}}$

Example 4: Solving Problems Using the Exponent Laws
A sphere has volume 425 m^3.
What is the radius of the sphere to the nearest tenth of a metre?

CHECK YOUR UNDERSTANDING
A cone with height and radius equal has volume 18 cm^3. What are the radius and height of the cone to the nearest tenth of a centimetre?
[Answer: approximately 2.6 cm]

Homework: Page 242 #7-10 odd letters, 11, 12, 15-17, 19